Probability Concept Map

Probability Concept Map

Alex Neill

Introduction **Language of probability**

Enumerating outcomes

Student misconceptions **Probability**

Qualitative probability

Concepts of probability **Quantitative probability**

Introduction

Probability is all around us. There are the weekly Lotto and Big Wednesday draws, poker machines proliferate, and many games are ruled by the roll of a dice, a coin toss, or the random numbers that computer games use. All of these are driven by chance (probability). Many other areas of our lives are increasingly being described in probabilistic terms. We hear that there is a 70% chance of rain, or that a court case has been decided beyond reasonable doubt (not with certainty you may note). Scientists say that they are 95% confident that a new medicine is effective, or doctors tell us that there is only a small chance of reacting negatively to a particular drug.
It can be debated whether independence, probability, and chance actually exists in the real world. Some people claim that given sufficient information that everything can be exactly predicted or specified. They are called "determinists" (see http://en.wikipedia.org/wiki/Determinism). Other people claim that the laws of probability do govern at least some physical events. These people follow what is known as "probabilism" (http://en.wikipedia.org/wiki/Probabilism). Whatever of these two viewpoints is held, probability is a highly effective way of modelling many real life situations.

Definitions of probability

- "The likely occurrence of an event (a particular outcome) which can be represented by a number from 0 (impossible) to 1 (certain)." (The National Standards publication, *Mathematics Standards for years 1–8*, p.52).
  They then define an outcome as "The result of a trial in a probability activity or a situation that involves an element of chance."
- A measure of how likely it is that some event will occur, which proportionately increases from 0 (impossible) to 1 (certain).

For a little more information about probability, go to:
http://www2.nzmaths.co.nz/frames/curriculum/glossary/glossary.aspx (the online glossary for the new curriculum) or
http://nzmaths.co.nz/probability-units-work

Relationship with statistical investigations and statistical literacy
Probability underpins the other two sub-strands of Statistics in the curriculum. Real-life data is variable, and statistics is the tool to help us cope with this variability. Probability is a way of describing and quantifying how things vary. So probability is not an end in itself, or something that only helps us to gamble, or play games of chance. It is a tool that helps us to investigate the real world we live in, and to respond critically to other people's statistical claims from investigations. Probability investigations also are one way to build up a feel for variability and distribution.
Probability investigations should follow the statistical enquiry cycle (SEC) outlined in the Mathematics and Statistics achievement objectives part of the New Zealand Curriculum. This involves:

- **Posing** a question to investigate.
- **Planning** how to do the investigation – also **Predict** what the results may be.
- **Data** collection.
- **Analyse** the data and form **conclusions**.
- **Communicate** the findings.

Two approaches to probability

- The *classical* or *frequentist* approach. These look at probabilities as the relative frequency of the occurrence of events. These are derived either from performing an experiment (where an *experimental* probability is found) or from enumerating all possible outcomes (where the *theoretical* probability is derived). Probabilities are usually expressed as fractions or decimal numbers, but also are expressed as percentages.
  The *frequentist* model covers **discrete** events (i.e. those with a set of distinct outcomes, such as the number on a dice) as well to situations where outcomes are **measurement** (continuous) data such as areas (or angles) on a spinner (e.g., Spinner probabilities).
  Students should have rich experiences in experimental probability before moving on to theoretical probabilities.

- The *bayesian* approach assigns probabilities to events that are based upon an individuals' degree of belief that something may happen. "The chance of rain today is 70%" fits into this category. This is not just the weather forecaster's blind faith, but their informed belief, and prior experience. A gambler may think "that horse has a good chance of winning" and go and place a bet on it.

Our curriculum and this concept map largely follow the *frequentist* model.

Language of probability

Here are some of the **words** we use to describe probability:
uncertainty, chance, risk, odds, likely, likelihood, possibility, luck, gamble, fair game, random or randomness, haphazard, chaotic, hit and miss.
These words are used to describe qualitative probability.
Some of these have particular technical meanings (for example relative *risk*, *odds* ratio, or *likelihood* function) but all are common ways of describing probability.

We **write** probabilities several different ways:

1. As a **decimal number**. The probability of getting an even number when we roll a dice is 0.5
2. As a **fraction**. The probability of getting an even number when we roll a dice is ½; **or**
3. As a **percentage**. The probability of getting an even number when we roll a dice is 50%.

Qualitative probability

This is where students give word descriptions about the probability of different things happening. For example, "It is *unlikely* that I find a $20 note today" or "I have a *good chance* of getting a phone call this evening". Often these descriptions cannot be given a numerical probability.

- Students should first be introduced to qualitative probability and the language of probability before quantifying it.
- Students then start relating qualitative probability to situations where the probability can be quantified.

Qualitative language
Here are some continuums:
- impossible, unlikely, even chance, likely, certain
- never, poor chance, even chance, good chance, always
- very rare, rare, very unlikely, very likely, almost always

0		0.5		1
Impossible	Unlikely	Even	Likely	Certain

Students sometimes believe that nothing is absolutely *impossible* nor is it completely *certain.* If this arises, talk about things that are impossible or certain by definition (e.g. "a normal dice that lands on a 7" or "a normal dice that lands on a number less than 7").

Resources

- Hot air balloon
- Counters in bag
- One dice chances
- Different spinners
- Is it possible?
- Possible events
- Describing probabilities
- Snakes and ladders
- Spinner chances
- Even Stevens
- Nuka Island
- Always, maybe, never
- What's the chance of that?

The definitions of qualitative are loose. Just how close to 0.5 does the probability need to be called an *even chance* instead of a *poor chance* (or a *good chance*)? When does it become *very likely* rather than just *likely*?

Quantitative probability

After working with qualitative probabilities, students need to move on to numerical ways of describing probabilities. This quantification needs to start young to help students progress. Probability usually is used to mean "quantitative probability".

***Counting outcomes from a fixed sample size*.**
For younger students this can be done by counting. If students perform an activity 10 times (for example), then they can count the number of times they were "successful". Ideas of variation can then be made by comparing these counts. This can move towards a fractional understanding when students say "Four out of 10 throws were heads".
Examples: Grab bag, Toy bag

Describing probabilities with fractions.
- Students at numeracy stages 4-5 should be ready to express probabilities as simple fractions, for example "the probability of getting tails is a half".
- At numeracy stage 6 they can progress onto other proper fractions, beginning with unit fractions.
Examples: Favourite All Black, Fruit in school

***Describing probabilities with fractions, decimals, and percentages*.**
- Sample sizes of 10 or 100 lend themselves naturally on to decimals and percentages (3/10 = 0.3, 31/100 = 0.31 which is 31%).
Examples: Favourite All Black, Fun run prizes, Two dice game IV
Moving between any probabilities in each of these three representations, and making comparisons with different sample size should be happening at numeracy stages 7-8.

Example: Missing cat.

Linking qualitative probability with quantitative probability

Qualitative description	Quantitative Probability
Impossible, never happens	0
Unlikely, poor chance	> 0 and <0.5
Even chance, 50/50	0.5 (½)
Likely, good chance	> 0.5 and < 1
Certain, always happens	1

***Describing probabilities with ratios*.**
Using ratios to describe probability is a Level 5 curriculum concept. This is referred to as the "odds" of something happening. Odds are often used in betting situations. Odds are slightly different from probability. Odds compare how probable something is of happening compared with it not happening. It is therefore a part:part relationship, rather than a part-whole one.
Example: The **odds** of getting a 3 with one roll of a fair dice is 1:5. There is 1 way of getting a 3, and 5 ways of getting another number. It is sometime written 1/5, though this is misleading because it can be confused with probability. The odds of getting a number 4 or less is 4:2, which is 2:1.
Using ratios is more an extension activity at curriculum levels 5-6. It only really comes into play at Level 7 with risk and relative risk. The statement "Men are fifteen times more likely that women to go to prison (i.e., the odds are 15:1)" is an example of *relative risk*, and is based on comparing ratios. This is an excellent situation to develop proportional reasoning.

Calculating quantitative probabilities

This is done by either:

i) The *experimental* probability = $\dfrac{\text{The number of "successes"}}{\text{The number of trials}}$

If we get 9 heads with 20 tosses of a coin, the *experimental* probability is 9/20 (= 0.45 or 45%).

ii) The *theoretical* probability = $\dfrac{\text{No. of "favourable" outcomes}}{\text{No. of all possible outcomes}}$

The *theoretical* probability of getting a 5 or 6 with a fair dice = 2/6 = 1/3 because there are 6 ways a dice can land (i.e. 6 possible outcomes), and only two of them (5 and 6) are "favourable outcomes" that is, they "are in favour" of the event we are getting a probability for. If we throw a fair dice a large number of times, the 5 or 6 should come up close to two-sixth (2/6) of the time.

In many situations, there is no theoretical probability. This is the case for throwing an unfair dice, or when you feed a ping pong ball into the clown’s mouth at the sideshow. The only way of estimating the actual probability of an event is to perform an experiment.

The number of trials used to determine the experimental probability is analogous to the *sample size* of a statistical survey.
The number of all possible outcomes is related to the concept *population* of a statistical survey.

Enumerating outcomes

Listing all possible outcomes helps to evaluate *theoretical* probabilities. This is particularly accessible when each of the outcomes are equally likely. Children can cope with very easy lists of possible outcomes which are discrete (such as "a coin lands on either heads or tails" or "a dice lands on 1, 2, 3, 4, 5, or 6"). The outcomes may also be measurement variables such as area, e.g., Prize wheel, or Different spinners. More complex situations that involve two or more variables situations can be described by looking at combinations.

- Acknowledge and anticipate possible outcomes (L1)
- List all possible outcomes (L2)
- Using models of all outcomes (L3-4). Models include:
  a) systematic lists (e.g., Numbers and letters,
  b) tree diagrams (click on link or keywords combinations AND tree diagrams),
  c) tables, or networks (see

Resources

- Combinations of soup and savouries
- Combinations of ties and socks
- Camp dinners
- Numbers and letters
- Clothing combinations V
- Dice combinations III
- Birthday combinations
- Dice combinations IV
- Draw of socks
- Scrabble tiles
- Interior decoration
- Lunchtime combinations
- Sandwich combinations II
- Desk combinations
- Snack food combinations
- Candle combos
- Combinations of marbles
- Coloured paper
- Friends combinations
- Sandwich combinations
- Clothing combinations I
- Clothing combinations II
- Dice combinations II
- Racing car

combos
- Yum Takeaways
- Coloured spinner
- Clothing combinations III
- Dice combinations I
- Dart combinations
- Table tennis
- Cups and saucers
- Coin throws

exemplars www.tki.org.nz/r/assessment/exemplars/maths/Stats_probability/sp_overview_e.php).

Concepts of probability

Variability
This is the idea that outcomes differ. This is true in both probability situations (e.g., dice games) as well as in statistical investigations (e.g., measurements of students' height).

Distribution
The distribution is a graph of the probabilities of events happening. Plotting the results of an experiment will give a picture of what this looks like. This is referred to as the ***experimental distribution***. Plotting the number of all possible outcomes gives the ***theoretical distribution***.
Example: If a dice is thrown 24 times, each throw may have a different number. These could be plotted as the experimental distribution. This could be compared with the theoretical distribution of 4 of each number coming up.

x x x **1** x x x x

x x x x x x **2** x x x x

x x x x **3** x x x x

x x **4** x x x x

x x x x x x x **5** x x x x

x x **6** x x x x

Experimental Distribution *Theoretical Distribution*

Click on the link to see a selection of resources about distribution.

Fairness – Fair games
This (in its simplest form) asks whether players in a game have an equal chance of winning. It could be extended to games where each player has different probabilities of winning but equal *expected return* on their money.

Click on the link to see a selection of resources about fair games.

Effect of sample size
The big idea is the larger the sample, the more accurate the *experimental probability* will be.

For example, a sample of 100 gives a more accurate estimate of the actual or theoretical probability than a sample size of 10. This is a very important concept. It can be explored by combining the results of classroom experiments. (e.g., Summer births, Five dice game I)

Common Student misconceptions

All resources with the keyword *probability* have the possibility of these misconceptions. The following resources give more detailed information about the misconceptions, and possible "Next steps" for learners: Spin a surprise, Roll a prize, Throwing a coin, Spinner probabilities.

Equiprobability
The student sees events as *equally likely* even when they have different probabilities. This involves either ignoring the different probabilities of events e.g., Marble bag or the number of different combinations e.g., Two dice game I

Examples:

1. If a coin were thrown two times, they would see 1 head and 1 tail as just as likely as 2 heads. They ignore that 1 head and 1 tail can occur 2 different ways, and is therefore twice as likely as 2 heads.
2. If a soccer team plays they can win, lose, or draw, so the probability of a draw must be 1/3.

Outcome approach
Some students may respond to situations involving chance by stating that you just can't tell anything when it comes to probability and may say something like, "It's all a matter of chance". This misconception sometimes looks like equiprobability as some of these students may say events are *equally likely* to happen regardless of any other factors. Such students may need to be questioned further to see if it is the *outcome* or the *equiprobability* misconception

Representative approach
Students select events that are most similar to the population that it comes from. They may also confuse the probability of a particular outcome with what they think best represents the overall population.
Examples:

1. A long run of consecutive heads in coin tosses is not representative of coin throws so is not likely.
2. A spread of lotto numbers is better than numbers that are too bunched up.
3. In six coin tosses there should be 3 heads and 3 tails because the probability is ½ (=3/6) of getting each one.
4. In 20 coin tosses, there will almost always be between 8 and 12 heads as it should be close to 10. The rules of probability show that the more likely range is 6 to 14, or even 5 to 15.

Negative regency (a form of representative bias)
Students often think that if a particular event has happened more often than they expect, then it will be less probable to occur.
Example:
A coin has come up heads lots more than tails, so it must be tails turn.

Lack of independence - influence
Students believe that previous events can influenced by a range of factors. For example the outcome of a coin toss may depend on:

- What the previous tosses were (see negative regency above).
- How the coin was tossed.
- What I want to happen.
- External, sometimes forces.

The belief that one event (e.g. throw of a coin) can be independent of the next is a basic presupposition of probability.
Some events are influenced by previous ones by definition. For example, If a card is taken from a deck of 52 cards, the probability of it being a specific card (say the two of clubs) is 1/52. If another card is now selected, then the probability of it being the two of clubs is then 1/51 (assuming that the first card was not the two of clubs). This is a simple example of **conditional probability**. This is not met until the senior secondary school.

Availability – belief
Students may have an underlying conviction that one particular outcome is more likely. It may be that "Heads" is their favourite (or that "3" is their favourite number), or that "6" is hard to roll on a dice. Some students may think that they can influence the outcome or that some external agency is determining the results.

Availability – experience
Students may be influenced by previous experiences. If they have recently found some money, they will think that it is far more likely to happen again than it is in reality.

Proportionality misconception – lack of part-whole thinking
Students think that the bigger the absolute size of something, the higher the probability.
Examples:

1. It is more likely to choose a red marble from a bag with 10 red and 20 blue ones than choosing one from a bag of 5 red and 10 blue ones. Cups and saucers (part b).
2. A larger spinner is more to come up with a win than a smaller spinner even when the relative areas are the same. Spinner probabilities.

Probability is the *relative frequency* of events or the *relative sizes* of measures (such as area), and not the absolute sizes. This is why it fits so well with work on fractions.

Promotional Text:
Alex Neill, 2008