Computational estimation concept map

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Introduction

Computational estimation is being able to quickly and easily get a number that is close enough to the exact answer of a mathematical problem to be useful. Usually it involves some simplified mental calculation. Every time you use a calculator, for example, it would be useful to know whether the answer it gives is sensible. This requires making an estimate. Sometimes a problem does not need an exact answer because the problem itself is not exact.

Computational estimation is:

- using some computation
- using easy mental strategies
- using number sense
- using a variety of strategies
- getting close to the exact answer

It is not:

- just a guess
- doing hand calculations
- using a calculator
- exact

Estimation takes forms other than just computational estimation. It includes estimation of the number of distinct objects in a set (referred to as numerosity) or estimation of the size of measurements such as height, weight, or area. This is referred to as measurement estimation. Statistics can also involve estimation about features of a population being studied, or about an experiment.

Background

This framework is based on a research project that looked at Year 8 students'

- understanding of, and
- ability with computational estimation.

It used information and ideas gathered from this research and the literature. As a result of this research a number of resources were developed and trialled with a small group of students. These resources have been placed onto the ARB website. Resources that are in bold in this document refer to ones that are a direct result of this research.

This framework is intended to be used by teachers to introduce computational estimation, and to give them a range of resources to assist in this. It provides information about what computational estimation is, student skills and attitudes that foster it, as well as a progression of teaching it. It includes a range of appropriate strategies, and links these to a directory of resources which utilise them. At present, the resources are focussed on estimation using whole numbers only.

Research

The literature indicates that certain skills and attitudes assist in becoming a good estimator.

Skills

- having good basic facts
- being able to do mental computation
- understanding of operations
- having place value knowledge
- having good number sense
- being able to work with powers of 10, especially in multiplication or division

Attitudes

- being confident and positive about mathematics
- recognising the usefulness of estimation
- being willing to accept a range of strategies
- being willing to accept a range of estimated values
- being a versatile thinker.

NOTE: Resources with an asterisk (*) have more detailed information about estimation, are linked to this document, and many use the methods outlined in expanding students' repertoire of estimation methods.

Estimation assessment

Who is estimating?
These resources can be used to assess whether students have an understanding of what estimation is and, to a lesser extent, which methods can be used to estimate.

When to estimate?
Students need to know in which situations it is appropriate to use estimation. These situations basically fall into three categories:

- There is no need to have an exact answer. An estimate is good enough: for example "Do I have enough money?"
- There is not enough information to get an exact answer: for example, "About how many times will my heart beat in an hour?"
- To check if the answer from a calculation is sensible.

Is it sensible?
Any time a student does an exact calculation, either with a calculator, by hand, or with a computer they need to be aware if the answer is sensible. This can be done in a wide variety of ways, including using many of the types of strategies referred to in the directory of estimation resources. Even simple tricks like knowing what is in the ones position of the answer to a multiplication problem is useful. Example: 53 × 246 must end in an 8 because. 3 × 6 = 18.

Order of magnitude
The most powerful way of deciding if the answer is sensible is by knowing the correct orderofmagnitude for it. Is the answer in the tens, hundreds, thousands, millions, etc? This requires knowledge of dealing with powers of ten under a variety of situations. In addition and subtraction this means using only the most significant digit (e.g. 2 346 + 1 472 + 25 ≈ 2 000 + 1 000 – ignore the 25). For multiplication and division problems, the laws of indices are the most general way of achieving this. However, an exploration of the effect of multiplying or dividing by 10, 100, 1000 etc. is a useful tool. Exploring Table 1 would be useful, including looking for patterns in it, and exploring why these patterns exist.

Table 1: The updated: ten-times table

×	10	100	1000	10 000	100 000
10	100	1000	10 000	100 000	1 000 000
100	1000	10 000	100 000	1 000 000	10 000 000
1000	10 000	100 000	1 000 000	10 000 000	100 000 000
10 000	100 000	1 000 000	10 000 000	100 000 000	1 000 000 000
100 000	1 000 000	10 000 000	100 000 000	1 000 000 000	10 000 000 000

Students cope with the order of magnitude in two different ways. These are best exemplified in addition.

- Extracted digits (EXT). Uses just one digit (the most significant one) in computation, then

expresses this in the correct order of magnitude.
Example: 23 717 + 54 834 ≈ 2 + 5 = 7 so it's 70 000.

- Same number of digits (SND). Holds all trailing zeros as placeholders.
  Example: 23 717 + 54 834 ≈ 20 000 + 50 000 = 70 000.
- The EXT method is generally preferable for multiplication.

Free estimation
These resources can be used to assess what strategies students use when given a variety of problems. A range of different strategies used by students is provided in the bolded resources.

Types of estimation

Three different types of computational estimation exist. The first is reformulation, which changes the numbers that are used to ones that are easy and quick to work with. The second is compensation, which makes adjustments that lead to closer estimates. These may be done during or after the initial estimation. Finally there is translation, which changes the mathematical structure of the problem (e.g. from addition to multiplication). Changing the form of numbers so that it alters the mathematical structure of the problem is also translation (Example: 26.7% of $60 requires multiplication, but this is about 1/4 of $60, which uses division).

Reformulation

Changing the numbers used. This is by far the most common type of strategy. It involves changing the numbers to ones that are more easily manipulated using mental strategies. Typically the original numbers are amended as tidy numbers (which are numbers that end with at least one 0). Rounding and front-end utilise tidy numbers.

Front-end (sometimes called truncation or rounding down)
This estimation strategy generally uses only the most significant (left-most) digit of the numbers being estimated. This strategy is most powerful when adding and multiplying. With these two operations, the exact answer is always underestimated. It is not as accurate as rounding, but is very easy to use, and makes compensation easier. Examples:

- 4 164 + 2 545 ≈ 4 000 + 2 000
- 41 × 27 ≈ 40 × 20

Rouding
This estimation strategy approximates the numbers being estimated to the nearest appropriate power of 10. Many students round inappropriately and still have to do calculations that they cannot do mentally (e.g. 35 × 85 is just as hard as 33 × 86). Numbers should be rounded to ones that can easily be computed mentally. Rounding may overestimate or underestimate the exact answer. It is often quite accurate, but compensation with rounding is often harder to use than with front-end estimation. Examples:

- 4 164 + 2 545 ≈ 4 000 + 3 000
- 41 × 27 ≈ 40 × 30

Rounding one number
In subtraction, just the smaller of the numbers needs to be rounded. This could also be done when adding, where all but the last number would be rounded. It is also sometimes useful in multiplication. Compensation is then easy to do. Examples:

- 4 164 – 2 745 ≈ 4 164 – 3 000
- 4 164 + 2 745 ≈ 4 164 + 3 000

- 37 × 96 ≈ 37 × 100 = 3 700

Rounding up
This is a form of compensation that ensures that the estimate is bigger than the exact answer. In many situations this is essential, for example in knowing if you have enough money for a purchase, or need a whole number answer to ensure you have enough of something.
Example:

- How many whole fish costing $4.15 each can be bought with $20?
- Round $4.15 up to $5, so you can buy 20 ÷ 5 = 4 fish (intermediate compensation).

Interval estimation
This strategy requires students to make a reasonable estimate of the range the answer must fall in. It will typically have a lower limit that the answer must exceed, and an upper limit, which will be larger than the exact answer. One way to achieve this is to apply both rounding and front-end strategies to give an interval between which the answer lies. The front-end estimate is always too small. If all numbers are rounded up, the estimate is always too large. Examples:

- 36 × 57
- To get the lower limit of the interval, use front-end: 30 × 50 = 1 500.
- To get the upper limit of the interval, use rounding up: 40 × 60 = 2 400.
- The answer lies between 1 500 and 2 400.

- 341 + 572
- To get the lower limit of the interval, use front-end: 300 + 500 = 800.
- To get the upper limit of the interval, use rounding: 400 + 600 = 1 000.
- The answer lies between 800 and 1 000.

Nice numbers (sometimes called Compatible Numbers)
This estimation strategy involves changing the numbers to be estimated to ones that have properties that make estimation easier. This is often done in conjunction with rounding or front-end estimation. Nice numbers use more than merely tidy numbers (i.e. ones that end in zeros). The numbers must be related to each other in some specific way. There are several variations of this strategy.

- Grouping nice numbers (within 10, 100, 1000)
  Group together numbers that sum to 10, 100, etc. Examples:
  8 + 2 = 10 so 83 + 23 ≈ 80 + 20 = 100
  43 + 17 = 60 so 437 + 174 ≈ 430 + 170 = 600
  In addition the numbers may occasionally be rounded to the nearest power of 10 for the most significant digit and to the nearest '5' for the second most significant digit. This is referred to as mid-rounding.
  Example: 443 + 362 ≈ 450 + 350. (because 50 + 50 = 100)

- Nice numbers and factors
  This is useful in division where the numbers to be estimated can be changed to ones that assist in division. This is done by changing the numbers, so that one of them is a factor of the other. These factors should be based upon basic multiplication facts.
  Examples:
  2 964 ÷ 7 ≈ 2 800 ÷ 7 = 400
  2 284 ÷ 59 ≈ 2 400 ÷ 60 = 40

- Changing one number
  This is useful when subtracting. Change all but the front-end digit of the smaller number to be the same as the larger one or vice versa.

Examples:
4 817 – 2 693 ≈ 4 817 – 2 817 = 2 000
4 817 – 2 693 ≈ 4 693 – 2 693 = 2 000
(In these examples the answer is underestimated and compensation can be used to get a closer estimate.)

Compensation

This includes either making adjustments before an approximate computation, or after an initial estimation to update it to a more accurate one.

Intermediate compensation (compensation during estimation or pre-compensation)
This form of compensation occurs when changing the original numbers, but before any approximate computation has been done. When multiplying two numbers, this can mean rounding one number up by a similar proportion to the amount that the other one has been rounded down (and not by similar absolute amounts). Examples:

- 40 × 30 is a close estimate to 43 × 28 as both numbers are rounded (one up, one down) by about the same amount. This only works if the two numbers are close together.
- 10 × 110 is a close estimate to 11 × 99 because 11 is rounded down by about 10% to 10, so 99 needs to rounded up by about 10% to 110.
- 173 + 282 + 368 + 189 + 572 ≈ 200 + 300 + 400 + 200 + 500 = 1 600. In this example 572 is rounded down to compensate for all the other numbers being rounded up.

Doubling and halving is a useful intermediate strategy in multiplication:
Examples:

- 14 × 26 ≈ 7 × 52 = 7 × 50
- 16 × 56 = 32 × 28 ≈ 30 × 30

Final compensation
This form of compensation occurs after the initial estimate is made. The estimate is updated to take account of about how far out the initial estimate is. The most critical information needed is to be able to tell if the initial estimate is too big or too small. The amount that it is over or under by is then estimated, and the initial estimate is then updated. Compensation is easier with front-end than with rounding, as all compensations involve adding to the original estimate, rather than a mix of addition and subtraction. Examples:

- 278 + 543 ≈ 300 + 500 = 800.
- This is over by about 30 but under by 40 so it's under by 10 so it's about 810. (Compensation after rounding)

- 278 + 543 ≈ 200 + 500 = 700.
- This is under by about 70 + 40 = 110 so it's about 810. (Compensation after front-end)

- 228 × 7 ≈ 200 × 7 = 1400.
- This is under by about 30 × 7 = 210 so it's about 1 600.

Doing front-end on the first digit and then rounding the second gives the best of both worlds.
Example:

- 278 + 543 ≈ 200 + 500 = 700.
- This is under by about 80 + 40 = 120 so it's about 820. (Compensation with rounding after front-end)

Translation

Changing the structure of the problem. This may change the arithmetic operation(s) being used. It includes changing the mathematical form of the number if this entails different arithmetic operators.

Averaging (also known as Clustering)
By observing that several numbers cluster about some average number, successive addition can be turned into multiplication. Example: 631 + 589 + 594 + 614 ≈ 4 × 600 = 2 400 as all 4 numbers are about 600.

Language of estimation

When doing estimation, use both the formal and the informal language of estimation.

- Some of the formal language is shown in the directory of estimation resources, and each of these specific techniques is discussed in more detail in the sections that follow. Terms such as front-end, rounding, interval, nice numbers, compensation, etc. should be used regularly so that they become embedded in students' mathematics vocabulary.
- The informal language of estimation should also be used. Use words and phrases such as: about, roughly, educated guess, good guess, guestimate, close to, thereabouts, something like, not far from, more or less, is near enough to, approximately.
  (NOTE: Some people distinguish between estimation and approximation. The latter is closing in on a particular target value, and having a way of knowing the limits of how close to the target you are).
  Avoid round about, as it reinforces the very common idea that estimation is merely rounding, rather than a rich range of strategies.
  Some informal language is best reserved for specific classes of estimation as the following table shows:

Table 2: Classroom language of estimation

Formal	Informal
Front-end	not quite, almost, a little less than, nearly, just about, take just the first number, cut-off (truncation)
Rounding	nearest to
Interval	between … and … , in the region of
Nice numbers	compatible, go together, well matched
Compensation	adjust, getting nearer, closer to, better, approaching, update, amend, improve, revise, modify your estimate, make it bigger, make it smaller
Averaging	grouped about … , all close to, clustered around

Expanding students' repertoire of estimation strategies

Our research indicated that many students (and teachers too) equate estimation with rounding. To counteract this, a number of resources have been developed to target specific estimation strategies. They are intended to introduce students to new ways of estimating that they may not have previously seen. These resources are structured so that students get a chance to do an estimation problem whatever way they wish, and then share each other's methods. If no students have used the target method, the teacher then introduces it, and the students then practice it on some examples. Some resources allow students to design and evaluate problems constructed by their peers.

The structure of these resources is as follows:

1. Do the estimation using any method.
2. Discuss the methods used.
3. Direct students to the target method.
4. Do the estimation using the target method.
5. Design a problem which could use the target method
6. Do a neighbour's problem. Could the target method be utilised by it?

Some estimation methods are more suitable than others are for using this structure. Resources which use all or some of this structure are listed in the resources column. All of these have been trialled on Level 4 (Year 8) students.

Resources using estimation with rational (fractional) or irrational numbers
The resources referred to in this directory relate mainly to estimation using whole numbers. Many of the methods used in the document above can be generalised to include rational numbers such as fractions, decimal fractions, and percentages, or irrational numbers such as square roots.

Introducing students to estimation

Some common ideas come through as things that a teacher should consider when using estimation with students:

- Discuss why estimation is important
- Value the role of estimation
- Find out where students use estimation and what they know about it
- Use real examples
- Use situations where an estimate is acceptable or essential
- Use the language of estimation
- Accept a range of estimates
- Discuss a range of strategies
- Share each others' strategies
- Expanding student's repertoire of estimation strategies through discussion and teaching
- Do examples that aren't too hard or too easy
- Do a little of it often
- Emphasise mental strategies.
- Sometimes limit the time students have to solve a problem to encourage estimation
- Link estimation with the reasonableness of exact calculations

Possible sequence of estimation teaching
What is estimation? Look for real examples of it in the media. Talk about what estimation is compared with exact computation.
Who is estimating?
When to estimate
Is it sensible?
Types of estimation.

A good sequence of strategies for estimation using whole numbers only is:

Addition	Use front-end, followed by rounding, then compensation especially with front-end. Next introduce grouping nice numbers (make to 10, 100, 1000 etc.).
Multiplication	Use front-end, followed by rounding. Then intermediate compensation and final compensation. Next introduce interval estimation. The averaging method turns an addition problem into multiplication. Look at the updated ten-times table, and order of magnitude for larger products (this may need to be regularly revisited).
Subtraction	Use rounding one number, then final compensation.
Division	Look at nice numbers and factors.

Directory of estimation resources (using whole numbers)

Estimation assessment	Resources
Who is estimating?	Who is estimating? Addition (addition) Who is estimating? Multiplication (multiplication)
When to estimate	TIMSS task Level 3 Question 2
Is it sensible?	Butter fingers (variety of operations)
Order of magnitude	Missing the point (variety of operations) Butter fingers (variety of operations) Number items B (part c) addition) TIMSS Level 4 Questions 8,12 & 19
Free estimation	Estimating lots (multiplication) Estimating sweets and buses (multiplication) Estimating addition (addition) Estimating costs (variety of operations) Estimate these (variety of operations) Estimate the menu (variety of operations) Estimate these II (variety of operations) Estimating stamps, money and pinecones (variety of operations)
Types of estimation	Resources
Front-end	Estimating sums of money (addition) Estimating stamps (multiplication)

Rounding	Estimating farm animals (addition) Estimating bags and boxes (multiplication) Estimating people (subtraction) Fish shop (multiplication) Estimating money (variety of operations)
Interval estimation	Estimating food numbers (multiplication) Square flower beds (square roots)
Nice Numbers	Estimating scores and crowds (addition) Estimating sweets (division)
Intermediate Compensation	How I estimate: Multiplication (multiplication) At the garden centre (variety of operations) Fish shop (multiplication)
Final compensation	Estimating team scores (addition) Estimating money (variety of operations) Fish shop (multiplication) At the garden centre (variety of operations)
Averaging	Estimating in sport (addition)

Promotional Text:
Alex Neill, 2004